Point vortices in a circular domain: stability, resonances, and instability of stationary rotation of a regular vortex polygon

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Abstract :

The paper is devoted to stability of the stationary rotation of a system of n equal point vortices located at vertices of a regular n-gon of radius R_0 inside a circular domain of radius R with a common center of symmetry. T. X. Havelock stated (1931) that the corresponding linearized system has an exponentially growing solution for $n \geq 7$, and in the case $2 \leq n \leq 6$ – only if parameter $p = R_0^2/R^2$ is greater than a certain critical value: $p_{*n} < p < 1$. In the present paper the problem on stability is studied in exact nonlinear formulation for all other cases $0 < p \leq p_{*n}$, $n = 2, \ldots, 6$. We formulate the necessary and sufficient conditions for $n \neq 5$. For the vortex pentagon it remains unclear the answer to the question about stability for a null set of parameter p. A part of stability conditions is substantiated by the fact that the relative Hamiltonian of the system attains a minimum on the trajectory of a stationary motion of the vortex n-gon. The case when its sign is alternating, arising for $n = 3, 5$, did require a special study. This has been analyzed by the KAM theory methods. Besides, here are listed and investigated all resonances encountered up to forth order. It turned out that two of them lead to instability.

Key-words :

vortices, stability, resonances

1 Introduction

In recent years, the model of point vortices has demonstrated its fundamental importance in studies of vortices in liquid helium and electron columns in plasma physics. The results obtained are stimulating new applications and additional studies of the model in different fields of research (experiment, theory, and numerical simulations) (see Aref *et al.* (2003), Borisov *et al.* (2003)). This has already led to the solution of several old problems. Recently, the efforts of many researchers were completed in a mathematically rigorous solution to the Kelvin problem (formulated in 1878) on the stability of steady-state rotation in a system of n identical point vortices located in a plane at apexes of a regular n-gon (see Kurakin and Yudovich (2002)): the rotation is stable only if $n \leq 7$, whereas, at $n \geq 8$, this motion is unstable. In the case of $n \neq 7$, linear analysis turns out to be sufficient to arrive at a conclusion concerning nonlinear stability. At the same time, for $n = 7$, it is necessary to involve nonlinear terms in the analysis.

The present paper deals with the Kelvin problem for the case in which the vortex n-gon of radius R_0 is located within circular domain R with a common center of symmetry. Havelock (1931) was the first to study this problem using mathematical methods in the linear formulation. As was shown in Havelock (1931), the corresponding linearized system has exponentially growing solutions at $n \geq 7$ and also in cases $(2 \leq n \leq 6)$ in which the parameter $p = \dfrac{R_0^2}{R^2}$ exceeds a certain critical value, $p_{*n} < p < 1$. In all other cases, the linear system exhibits only a power-law instability, which is usual—and inevitable—for systems of such a kind.

According to the well-known Lyapunov theorem, the equilibrium of a complete system is unstable when a linearized system is exponentially unstable. The power-law instability is insufficient to draw this conclusion; therefore, nonlinear terms should be involved in the analysis.

The substantiation of all the conditions for nonlinear stability on a plane and sphere in the Kelvin problem is based on the fact that, in the trajectory of steady-state motion for the vortex n-gon, the relative Hamiltonian (see, for example, Kurakin and Yudovich (2002)) attains its maximum value. In this case, the stability of steady-state motion is treated as stability in the Routh sense.

In the present paper, such an approach allows us to prove the stability of steady-state rotation of the regular vortex n-gon within a circle. The proof is obtained in the exact nonlinear formulation of the problem in the cases (a) $0 < p \le p_{*n}$ for even n ($n = 2, 4, 6$), (b) $0 < p < p_{03}$ for three vortices, and (c) $0 < p \le p_{05}$ when $n = 5$.

$P_2 = 7p^3 - 3p^2 + 5p - 1$	$p_{*2} \approx .2137403629$
$P_3 = 10p^6 + 3p^5 + 6p^4 + 10p^3 + 6p^2 + 3p - 2$ $Q_3 = 5p^6 + 9p^5 + 5p^3 + 9p^2 - 1$	$p_{*3} \approx 0.3212811546$ $p_{03} \approx 0.3040641646$
$P_4 = 7p^6 + p^4 + 9p^2 - 1$	$p_{*4} \approx .3298399891$
$P_5 = 18p^{10} + 10p^8 + 15p^7 + 34p^5 + 15p^3 + 10p^2 - 2$ $Q_5 = 27p^{12} + 81p^{11} + 132p^{10} + 135p^9 + 90p^8 + 96p^7 + 153p^6 + 196p^5 + 165p^4 + 60p^3 + 2p^2 - 9p - 3$	$p_{*5} \approx 0.3461008645$ $p_{05} \approx .3410383818$
$P_6(x) = 23p^9 + 13p^6 + 37p^3 - 1$	$p_{*6} \approx .2991212951$

Table 1
Critical values p_{*n} and p_{0n}, which are the roots of the polynomials P_n and Q_n, respectively

The numerical analysis performed in Campbell (1981) revealed the alternating-sign behavior of the relative Hamiltonian under conditions (d) $n = 3$, $p_{03} < p < p_{*3}$ or (e) $n = 5$, $p_{05} < p < p_{*5}$, although, in these cases, the linearized system is characterized only by the power-law instability. Such a situation is analyzed below using the methods of the Kolmogorov–Arnold–Mozer theory. In addition, we list and analyze all resonances up to the fourth-order ones available in the system. This analysis is based on the results of Markeev and Sokol'skii (see Markeev (1978)). It turned out that two of these resonances led to instability: (f) $n = 3, p = p_{*3}$ and (g) $n = 5, p = p_{*05} \approx .3443792197$.

As a result, we present in this paper both the necessary and sufficient conditions for the stability and instability of a regular n-gon of point vortices ($n \neq 5$) located within a circle. For a vortex pentagon, the answer to the question concerning the instability has remained unclear for the null set when the parameter p meets conditions $p \in [a, b] \subset (p_{05}, p_{*5})$ and there exist resonances higher than four ($a \approx .3412172781, b \approx .3429140261$).

Some results of the present work were briefly reported in Kurakin (2004, 2005).

2 Equations of motion

The basic results on the motion of point vortices inside and outside the circular domain were systematized by Kilin *et al.* (Borisov *et al.*, 2003, pp. 414–440).

Motion of the system of n point vortices at the plane inside a circle of radius R is described by the equations

$$\dot{z}_k^* = \frac{1}{2\pi i}\sum_{j=1}^{n}{}' \frac{\kappa_j}{z_k - z_j} - \frac{1}{2\pi i}\sum_{j=1}^{n}\frac{\kappa_j}{z_k - \widehat{z}_j}, \quad k = 1, \ldots, n. \tag{1}$$

Here, $z_k = x_k + iy_k, k = 1, \ldots, n$ are complex variables; x_k, y_k are the Cartesian coordinates of the kth vortex; κ_k is its intensity; and $\widehat{z}_k = \dfrac{R^2}{z_k^*}$ is the reflection of the kth vortex from the circle boundary. The prime denotes the omission of the term with $j = k$, and the asterisk implies complex conjugation. The phase space Z for the set of Eqs. (1) is $(\mathbf{C} \setminus \{0\})^n$ with cuts along all the hyperplanes $z_j = z_k$, $j \neq k$.

The set of Eqs. (1) is the Hamiltonian set characterized by the Hamiltonian

$$H = -\frac{1}{4\pi} \sum_{1 \le j < k \le n} \kappa_j \kappa_k \ln[(z_j - z_k)(z_j^* - z_k^*)] + \frac{1}{8\pi} \sum_{j=1}^{n} \sum_{k=1}^{n} \kappa_j \kappa_k \ln[(R^2 - z_j z_k^*)(R^2 - z_j^* z_k)]. \quad (2)$$

It has two integrals: energy H and the total moment of inertia, $M = \sum_{k=1}^{n} \kappa_k |z_k|^2$.

This set is invariant with respect to the group G, with the group generators being the mirror reflection j: $z \mapsto z^*$ and rotation g^{rot}: $z \mapsto e^{i\alpha} z$, $\alpha \in \mathbf{R}$. The action $g \mapsto L_g$ of the group G on the phase space Z is determined by the relationship $L_g z = (gz_1, \ldots, gz_n)$ for an arbitrary point $z = (z_1, \ldots, z_n) \in Z$ and arbitrary motion $g \in G$.

It is worth recalling that the motion is referred to as steady-state motion if it is generated by transformations corresponding to a certain one-parameter subgroup of the symmetry group characterizing the equation under study.

We seek the steady-state motion corresponding to the subgroup of rotations g^{rot}, in the form $z_k = e^{i\omega t} u_k$. Then, the equation determining the steady-state motions is written as the following set of equations:

$$-i\omega u_k^* = \frac{1}{2\pi i} \sum_{j=1}^{n}{}' \frac{\kappa_j}{u_k - u_j} - \frac{1}{2\pi i} \sum_{j=1}^{n} \frac{\kappa_j}{u_k - \widehat{u}_j}, \quad k = 1, \ldots, n, \quad (3)$$

with respect to the unknowns $u_1, \ldots, u_n \in \mathbf{C}$ and $\omega \in \mathbf{R}$.

In the case of equal intensities $\kappa_1 = \ldots = \kappa_n = \kappa$, the exact solution to the set under consideration is well known:

$$u_k = R_0 e^{2\pi i (k-1)/n}, \quad k = 1, \ldots, n, \quad (4)$$

$$\omega = \frac{\kappa}{4\pi R_0^2} \left(\frac{n+1}{2} - \frac{n}{1 - p^n} \right), \quad (5)$$

where we have introduced the notation $p = \dfrac{R_0^2}{R^2}$ and R_0 meets the inequality $0 < R_0 < R$.

The corresponding steady-state mode is determined by the relationships

$$z_k(t) = R_0 e^{i\omega t} u_k, \quad k = 1, \ldots, n. \quad (6)$$

Thus, the configuration of identical vortices located at the circumference of radius R_0 at the apexes of a regular n-gon rotates at the constant angular velocity $\omega = \omega(p)$.

3 The stability of a regular vortex n-gon

We now assume all vortices to have the same intensity κ and analyze the stability of stead-state solution (6). For convenience of calculation, we consider that $R_0 = 1$.

The change of variables

$$z_k(t) = e^{i\omega t} v_k(t)$$

in set (1) results in the following equation describing the relative motion:

$$\dot{v}_k^* = \frac{1}{2\pi i}{\sum_{j=1}^{n}}' \frac{\kappa}{v_k - v_j} - \frac{1}{2\pi i}\sum_{j=1}^{n}\frac{\kappa}{v_k - \widehat{v}_j} + i\omega v_k^*, \quad k = 1, \ldots, n, \tag{7}$$

with the relative Hamiltonian

$$E(v) = H(v) + \omega M(v), \quad M = \kappa \sum_{k=1}^{n} |v_k|^2, \tag{8}$$

where $v = (v_1, \ldots, v_n) \in C^n$.

In each plane of variables v_k, we introduce new coordinates and write v_k in the form

$$v_k = \sqrt{2(\frac{R_0^2}{2} + r_k)}e^{i(\frac{2\pi}{n}(k-1)+\theta_k)}. \tag{9}$$

In variables $r = (r_1, \ldots, r_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$, Eq. (7) takes the form

$$\dot{r}_k = \frac{\partial E}{\partial \theta_k}(v(r,\theta)), \qquad \dot{\theta}_k = -\frac{\partial E}{\partial r_k}(v(r,\theta)). \tag{10}$$

Steady-state motion (6) is put in correspondence with the continuous family of equilibrium states of the set of Eqs. (10) located in the straight line $\Gamma = \{(r,\theta) \in R^{2n} : r = 0, \theta_1 = \ldots = \theta_n\}$.

The expansion of function $E(v(\rho)), \rho \stackrel{\text{def}}{=} (r,\theta)$ into the Taylor series has the same form in the neighborhood of each equilibrium state belonging to family Γ:

$$E(v(\rho)) = \frac{\kappa^2}{4\pi}(E_0 + E_2(v(\rho)) + E_3(v(\rho)) + E_4(v(\rho)) + \ldots). \tag{11}$$

Here, dots denote terms of the power higher than four. The quadratic form E_2 can be represented as

$$E_2 = (S\rho, \rho), \qquad S = \begin{pmatrix} F_1 & \frac{1}{2}G_0 \\ -\frac{1}{2}G_0 & F_2 \end{pmatrix}, \quad \rho = (r,\theta), \tag{12}$$

and the linearization matrix for the set of Eqs. (10) has the following form for the zeroth equilibrium state:

$$L = \begin{pmatrix} G_0 & 2F_2 \\ -2F_1 & -G_0 \end{pmatrix}. \tag{13}$$

Here, F_1 and F_2 are symmetric matrices and G_0 is a skew-symmetric matrix. Their matrix elements and eigenvalues $\lambda_{1k}, \lambda_{2k}$, and $i\lambda_{0k}, k = 1, \ldots, n$, were written out by Havelock (1931):

$$\lambda_{1k} = -\frac{1}{2}k(n-k) - (n+1) - \frac{n^2p^{n-k}(1+p^k)^2}{2(1-p^n)^2} - \frac{nk(p^k - p^{n-k})}{2(1-p^n)} + \frac{2n}{1-p^n}, \tag{14}$$

$$\lambda_{2k} = \frac{1}{2}k(n-k) - \frac{nk(p^k - p^{n-k})}{2(1-p^n)} - \frac{n^2p^{n-k}(1-p^k)^2}{2(1-p^n)^2}, \tag{15}$$

$$\lambda_{0k} = \frac{nk(p^k + p^{n-k})}{1-p^n} - \frac{n^2p^{n-k}(1-p^{2k})}{(1-p^n)^2}. \tag{16}$$

The eigenvalues of matrix S can be found using the roots of the polynomials,

$$\Lambda^2 - (\lambda_{1k} + \lambda_{2k})\Lambda + \lambda_{1k}\lambda_{2k} - \frac{1}{4}\lambda_{0k}^2, \quad k = 1, \dots, n.$$

For the linearization matrix L, the eigenvalues are determined by the formulas Havelock (1931)

$$\sigma_k^{\pm} = -i\lambda_{0k} \pm 2\sqrt{-\lambda_{1k}\lambda_{2k}}, \quad k = 1, \dots, n. \tag{17}$$

The following theorem validates the linearization method in the stability problem for a vortex n-gon. The stability in the Routh sense for steady-state solution (6) implies the stability of family Γ of equilibrium states for Eq. (7) corresponding to the relative motion. The instability is understood here in the strongest sense: the invariant set of steady-state rotations is (transversally) unstable. The values of p_{0k} and p_{*n} are specified in Table 1.

Theorem 1. *Steady-state rotation (6) of the regular vortex n-gon is stable in the Routh sense in the cases*

1°. $0 < p < p_{*n}$ *and for even values of* n *(*$n = 2, 4, 6$*),*

2°. $0 < p < p_{03}$ *for* $n = 3$*, and*

3°. $0 < p < p_{05}$ *when* $n = 5$*,*

as well as at $n = 1$*. It is unstable when* $n \geq 7$ *or* $p_{*n} < p < 1$ *at an arbitrary* $n = 2 - 6$*.*

The proof repeats the substantiation of the validity of linearization in the problem of stability for a regular vortex n-gon in a plane (see Theorem 3.1 in Kurakin and Yudovich (2002)).

The following theorem requires for its proof taking into account the nonlinear terms present in the system. Formal stability in the Routh sense implies that there exists a power series—possibly diverging—that is formally an integral of the equation with respect to relative motion (7). This integral attains the minimum value on the family Γ of equilibrium states for this motion. In the case of formal stability in the Routh sense, instability in the Lyapunov sense for family Γ (if it exists) does not manifest itself in the system even if, in the expansion, we take into account terms of an arbitrarily high (but finite) order. Below, we use the values $p_{0*5} \approx .3443792197$, $a \approx .3412172781, b \approx .3429140261,$ and an arbitrary nonzero vector (n_1, n_2, n_3, n_4) with integer nonnegative components.

Theorem 2. *Steady-state rotation (6) of a regular vortex n-gon is stable in the Routh sense in the cases*

4°. $p = p_{*n}$ *at an arbitrary* $n = 2, 3, 4, 6$ *and also at* $p = p_{05}$ *for* $n = 5$*;*

5°. $p_{03} < p < p_{*3}$ *for* $n = 3$*; and*

$0 < p < p_{05}$ *when* $n = 5$*,*

or formally stable in the Routh sense if

6°. $n = 5, p \in (p_{05}, a) \bigcup (b, p_{0*5}) \bigcup (p_{0*5}, p_{*5}]$ *or if* $p \in [a, b]$ *under the conditions* $n_1\sigma_1^+ + n_3\sigma_3^+ + n_4\sigma_4^+ \neq n_2\sigma_2^-$*.*

It is unstable in the following resonance cases:

7°. $n = 3, p = p_{03}$,

8°. $n = 5, p = p_{0*5}$.

Proof. First of all, owing to the existence of the cyclic variable, we reduce by one the number of degrees of freedom for the Hamiltonian system under study. Then, in the resonance cases listed in Table 2, we apply the appropriate theorems of Markeev and Sokol'skii (see Markeev (1978)). If the resonances are absent and inequalities 5° are met, stability is substantiated by verifying the validity of the conditions imposed by the Alnold–Mozer theorem (see Arnold (1963); Mozer (1973)), while, in case 6°, we use the theorems of Birkhoff, Glimm, and Brunot.

n=2	$\hat{p}_{00} = p_{*2}$
n=3	$p_{00} = p_{03}, p_{1:3} \approx .3168967611$ $p_{1:2} \approx .3193266263, \hat{p}_{1:1} = p_{*3}$
n=4	$\hat{p}_{00} = p_{*4}$
n=5	$p_{00} = p_{05}$ $p_{1:3} \approx .3434991204, p_{1:3} \approx .3448097395$ $p_{1:2} \approx .3443792197, p_{1:2} \approx .3455248914$ $p_{1:1} \approx .3459139152, \hat{p}_{1:1} = p_{*5}$
n=6	$\hat{p}_{00} = p_{*6}$

Table 2
Critical values of parameter p corresponding to resonances: p_{00} is the double diagonalizable zero and $p_{k:m}$ is the $k : m$ resonance; $\hat{p}$ denotes a nondiagonalizable case

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